# Generalized symplectic geometry as a covering theory for the Hamiltonian theories of classical particles and fields

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## Abstract

It is shown that the symplectic potentials which underlie the symplectic geometry of classical particles and the multisymplectic geometry of classical fields are obtained as projections of a generalized symplectic potential defined on appropriate bundles of frames on a certain manifold. In this sense generalized symplectic geometry is a covering theory for Hamiltonian theories of both particles and fields.

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## 1. Introduction

In this paper, we report on recent work in the development of a new geometrical theory that captures the essence of Hamiltonian methods for both particles and fields. The new geometry, $n$-symplectic geometry, is a generalization of standard symplectic geometry on the cotangent bundle of an $n$-dimensional manifold. In $n$-symplectic geometry, linear frame bundles behave as generalized phase spaces. Our goal is to demonstrate that $n$-symplectic

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geometry is a covering theory for standard Hamiltonian theories in the sense that the symplectic structures of standard theories are derived from the generalized symplectic structures of $n$-symplectic geometry. Moreover, we will show that the algebraic structure of observables in $n$-symplectic geometry is rich enough to resolve some outstanding difficulties in the multisymplectic geometry for classical fields.

## 2. $n$-symplectic geometry on the linear frame bundle

In this section, we summarize Norris's theory of $n$-symplectic geometry on the linear frame bundle [11]. Let $M$ be an $n$-dimensional manifold and let $\tau : LM \to M$ be the bundle of linear frames of $M$. That is,

$$LM := \{(x, e_i) \mid x \in M, \{e_i\} \text{ is a frame of } T_x M\}.$$

The structure group of $LM$ is the general linear group $GL(n, \mathbb{R})$, which acts freely on the right of $LM$. The frame bundle $LM$ supports a globally defined $\mathbb{R}^n$-valued one-form, the *soldering one-form* $\theta$, defined by [8]

$$\theta(Y) := u^{-1}(\tau_* Y) \quad \forall Y \in T_u LM,$$

where $u = (x, e_i) : \mathbb{R}^n \to T_{\tau(u)}M$ is the linear isomorphism $\xi^i r_i \mapsto \xi^i e_i$. Here and in the following, $(r_i)$, $i = 1, 2, \ldots, n$, denotes the standard basis of $\mathbb{R}^n$. When convenient we will express $\theta$ as $\theta^i r_i$ where each of the $n$ one-forms $\theta^i$ is real valued. Compare this $\mathbb{R}^n$-valued soldering one-form to the real-valued canonical one-form $\vartheta$ on the cotangent bundle $T^* M$.

Consider now the exact $\mathbb{R}^n$-valued two-form $d\theta$. It is straightforward to show that $d\theta$ is nondegenerate in the sense that

$$X \lrcorner\, d\theta = 0 \Leftrightarrow X = 0, \tag{1}$$

where $\lrcorner$ indicates the inner product [1] of a form with a vector field. Thus $d\theta$ has the basic properties of a symplectic structure, although it is $\mathbb{R}^n$-valued. This motivates the following definition.

**Definition 1.** Let $P$ be a principal fiber bundle over a manifold $M$ with group $G$. Let the dimension of $M$ be $n$. An *$n$-symplectic structure* on $P$ is an $\mathbb{R}^n$-valued two-form $\omega$ on $P$ that is closed and nondegenerate in the sense of Eq. (1). The pair $(P, \omega)$ is an *$n$-symplectic manifold*.

The theory of $n$-symplectic geometry [11] on $(LM, d\theta)$ is based on the generalized structure equation

$$d\hat{f}^{i_1 i_2 \cdots i_p} = -p! X_{\hat{f}}^{i_1 i_2 \cdots i_{p-1}} \lrcorner\, d\theta^{i_p}, \tag{2}$$

where the functions $\hat{f}^{i_1 i_2 \cdots i_p}$ are the components of a $\otimes^p \mathbb{R}^n$-valued function $\hat{f}$, and the vector fields $X_{\hat{f}}^{i_1 i_2 \cdots i_{p-1}}$ are the components of a $\otimes^{p-1} \mathbb{R}^n$-valued vector field $X_{\hat{f}}$. Recall

that the soldering one-form $\theta$ transforms tensorially under right translations $R_g$ according to $R_g^* \theta = g^{-1} \cdot \theta$ for each $g \in GL(n, \mathbb{R})$. A consequence of this tensorial nature of $\theta$ is that not every $\otimes^p \mathbb{R}^n$-valued function $\hat{f}$ is compatible with the structure equation (Eq. (2)), and hence $n$-symplectic geometry selects classes of allowable observables. This is in contrast with the fact that all smooth real-valued functions on $T^*M$ are allowable observables.

The allowable observables in $n$-symplectic geometry divide naturally into the symmetric and antisymmetric Hamiltonian functions, which we denote by $SHF$ and $AHF$, respectively. The space $SHF$ is the direct sum $\bigoplus_{p=1}^{\infty} SHF^p$ where $SHF^p$ is the space of $(\otimes_s)^p \mathbb{R}^n$-valued functions defined on $LM$ that are compatible with a symmetrized version of (2), and $\otimes_s$ denotes the symmetric tensor product. The elements of $SHF^p$ are, in local canonical coordinates on $LM$, $p$th degree polynomials in the generalized momentum coordinates on $LM$, with coefficients that are constant on the fibers of $LM$. In particular, those elements of $SHF^p$ whose local coordinate representatives are homogeneous polynomials are in one–one correspondence with symmetric rank $p$ contravariant tensor fields on the base manifold $M$. Since those are precisely the elements of $SHF^p$ which are tensorial we denote this subset by $ST^p$ (for Symmetric and Tensorial) and set $ST = \bigoplus_{p=1}^{\infty} ST^p$. We note in particular that elements of $ST^1$ are in one–one correspondence with vector fields on $M$.

For the symmetric algebra $ST$, Eq. (2) is replaced by

$$\mathrm{d}\hat{f}^{i_1 i_2 \ldots i_p} = -p! X_{\hat{f}}^{(i_1 i_2 \ldots i_{p-1}} \lrcorner \mathrm{d}\theta^{i_p)}, \tag{3}$$

where the parenthesis on indices denote symmetrization over the enclosed indices. For each $\hat{f} \in ST^p$ this equation determines an equivalence class of $\binom{n+p-2}{p-1}$ vector fields for $p \geq 2$ because the symmetrization on indices in (3) introduces a certain degeneracy which is not present for $p = 1$. More precisely, for each $\hat{f} \in ST^p$ and each $x \in LM$ there exists a local section $X_{\hat{f}} = X_{\hat{f}}^{i_1 \ldots i_{p-1}} (r_{i_1} \otimes_s \cdots \otimes_s r_{i_{p-1}})$ of the vector bundle $T(LM) \otimes (\otimes_s)^{p-1} \mathbb{R}^n \to LM$ defined on an open subset $U_x$ of $LM$ such that the vector fields $X_{\hat{f}}^{i_1 \ldots i_{p-1}}$ satisfy (3) for all $(i_1, i_2, \ldots, i_p)$. In fact there exists more than one such local section since

$$K_y = \{Y_y \mid Y_y^{(i_1 \ldots i_{p-1}} \lrcorner \mathrm{d}\theta^{i_p)} = 0 \text{ for all } (i_1, i_2, \ldots, i_p)\}$$

is a nontrivial subspace of $T_y(LM) \otimes (\otimes_s)^{p-1} \mathbb{R}^n$ for each $y \in LM$. Indeed, $K = \bigcup_{y \in LM}(K_y)$ is a vector sub-bundle of $T(LM) \otimes (\otimes_s)^{p-1} \mathbb{R}^n$ and for each $\hat{f} \in ST^p$ there exists a unique global section $\sigma$ of $[T(LM) \otimes (\otimes_s)^{p-1} \mathbb{R}^n]/K \to LM$ such that for each $x \in LM$ there exists a local section $X_{\hat{f}}$ of $[T(LM) \otimes (\otimes_s)^{p-1} \mathbb{R}^n] \to LM$ having the property that $X_{\hat{f}}$ satisfies (3) and $\sigma(y) = X_{\hat{f}}(y) + K_y$ for each $y \in U_x$. We denote $\sigma$ by $[\![X_{\hat{f}}]\!]$. If we fix $i = (i_1, \ldots, i_{p-1})$ then there is also a sub-bundle $K^i$ of $T(LM)$ such that for $y \in LM$

$$K_y^i = \{Y_y^{i_1 \ldots i_{p-1}} \mid Y_y^{(i_1 \ldots i_{p-1}} \lrcorner \mathrm{d}\theta^{i_p)} = 0 \text{ for all } i_p\}.$$

Moreover there is a unique section $\sigma^i$ of $T(LM)/K^i \to LM$ such that for $y \in LM$, $\sigma^i(y) = X_{\hat{f}}^i(y) + K_y^i$ for some local section $X_{\hat{f}}^i$ of $T(LM) \to LM$ which satisfies (3)

for all $i_p$. We denote this section $\sigma^i$ by $[\![X_{\hat{f}}]\!]^i = [\![X_{\hat{f}}]\!]^{i_1\cdots i_{p-1}}$. The fact that one obtains equivalence classes of vector fields rather than vector fields for the higher rank observables does not interfere with the basic algebraic structures in $n$-symplectic geometry. For each $p \geq 1$ the set of equivalence classes of $\otimes_s^{p-1}\mathbb{R}^n$-valued vector fields on $LM$ forms an infinite-dimensional vector space. Denote by $HV(ST^p)$ the vector space of $\otimes_s^{p-1}\mathbb{R}^n$-valued equivalence classes of vector fields determined by elements of $ST^p$ using Eq. (3). For $\hat{f} \in ST^p$ and $\hat{g} \in ST^q$ define the Poisson bracket $\{\,,\,\} : ST^p \times ST^q \to ST^{p+q-1}$ by

$$\{\hat{f}, \hat{g}\}^{i_1 i_2 \cdots i_{p+q-1}} = p! X_{\hat{f}}^{i_1 i_2 \cdots i_{p-1}} (\hat{g}^{i_p i_{p+1} \cdots i_{p+q-1}}), \tag{4}$$

where $X_{\hat{f}}^{i_1 i_2 \cdots i_{p-1}}$ is any representative of the equivalence class $[\![X_{\hat{f}}]\!]^{i_1 i_2 \cdots i_{p-1}}$. The bracket so defined is easily shown to be independent of the choice of representatives and has all the properties of a Poisson bracket. In fact when the bracket defined here is re-expressed on the base manifold $M$, it gives [11] the differential concomitant of Schouten [13] and Nijenhuis [10] of the symmetric tensor fields corresponding to $\hat{f}$ and $\hat{g}$.

**Theorem 1.** *The space $ST$ of symmetric tensorial functions on $LM$ is a Poisson algebra with respect to the Poisson bracket defined in* (4).

One can also show [11,12] that there is a naturally defined Lie bracket, defined using equivalence class representatives, that is antisymmetric and satisfies

$$[\![X_{\hat{f}}]\!], [\![X_{\hat{g}}]\!]] = [\![X_{\{\hat{f}, \hat{g}\}}]\!]. \tag{5}$$

Denote the direct sum of the vector spaces $HV(ST^p)$ by $HV(ST)$.

**Theorem 2.** *The vector space $HV(ST)$ of vector-valued equivalence classes of Hamiltonian vector fields on $LM$ is a Lie algebra with respect to the naturally defined bracket.*

For later reference we make explicit the formulas for a rank one tensorial observable. Such an observable $\hat{f}$ corresponds [8] to a unique vector field $f$ on the base manifold $M$. Let $\{x^i\}$ be a coordinate chart on $U \subset M$. In local coordinates $\{x^i, \pi_k^j := e^j(\partial/\partial x^k)\}$ on $\tau^{-1}(U) \subset LM$, we may express $\hat{f}^i$ as $\hat{f}^i = f^j(x)\pi_j^i$ where $f = f^i(x)(\partial/\partial x^i)$, and $x \in M$. Eq. (3) now reduces to

$$d\hat{f}^i = -X_{\hat{f}} \lrcorner \, d\theta^i, \tag{6}$$

and this equation has a unique solution for $X_{\hat{f}}$, given in local coordinates by

$$X_{\hat{f}} = f^i(x)\frac{\partial}{\partial x^i} - \frac{\partial f^i}{\partial x^j}\pi_i^k\frac{\partial}{\partial \pi_j^k}. \tag{7}$$

This vector field is also known [8] as the *natural lift* of the vector field $f$ to $LM$. The reader is referred to [11] for a discussion of the full Poisson algebra $SHF$ and a discussion of the super-Poisson algebra $AHF$.

## 3. Generalized symplectic geometry on associated tensor bundles

Consider the vector space $(\otimes_a)^p \mathbb{R}^{n*}$ of rank $p$ covariant skew-symmetric tensors over $\mathbb{R}^n$, for $1 \leq p \leq n$, where $\otimes_a$ denotes the antisymmetric tensor product. With the usual $GL(n, \mathbb{R})$-action on $(\otimes_a)^p \mathbb{R}^{n*}$, we may form the associated tensor bundle $\wedge^p M \simeq LM \times_{GL(n,\mathbb{R})} (\otimes_a)^p \mathbb{R}^{n*}$ and denote the projection map by $\pi_p : \wedge^p M \to M$. This bundle possesses a *canonical p-form* ${}^p\Theta$ defined by ${}^p\Theta_{[u,T]} := \pi_p^*[u, T]$ where $u \in LM$ and $T \in (\otimes_a)^p \mathbb{R}^{n*}$.

Let $\theta$ be the canonical soldering one-form on $LM$. Define the $(\otimes_a)^p \mathbb{R}^n$-valued $p$-form $\wedge^p \theta$ by $\wedge^0 \theta := 1$ and

$$\wedge^p \theta := \overbrace{\theta \wedge \cdots \wedge \theta}^{p} \quad \text{if } p \geq 1, \tag{8}$$

where the $\wedge$ operates in both the domain and range of $\theta$. We may relate the canonical soldering one-form on $LM$ to the canonical $p$-form on $\wedge^p M$. Indeed, let $\hat{X}_i \in T_u LM$ and $\tilde{X}_i \in T_{[u,T]}(\wedge^p M)$ satisfy $\tau_* \hat{X}_i = \pi_{p*} \tilde{X}_i$ for $1 \leq i \leq p$. Then

$$\tilde{X}_p \lrcorner \cdots \lrcorner \tilde{X}_1 \lrcorner {}^p\Theta_{[u,T]} = \langle \hat{X}_p \lrcorner \cdots \lrcorner \hat{X}_1 \lrcorner \wedge^p \theta_u, T \rangle, \tag{9}$$

where the brackets denote the natural inner product of elements of $(\otimes_a)^p \mathbb{R}^n$ and $(\otimes_a)^p \mathbb{R}^{n*}$. If we fix $T$ and define the map

$$\phi_T : LM \to \wedge^p M : u \mapsto [u, T], \tag{10}$$

then we may write the relationship in (9) as

$$\phi_T^*({}^p\Theta) = \langle \wedge^p \theta, T \rangle. \tag{11}$$

Observe that $\phi_T$ is not surjective since the subset of $\Lambda^p M$ onto which $\phi_T$ maps depends on the rank of $T$.

If $1 \leq p < n$ then the $(p + 1)$-forms $d(\wedge^p \theta)$ and $d({}^p\Theta)$ are both nondegenerate. Thus the generalization of a symplectic structure on $\wedge^p M$ is directly related to the $n$-symplectic structure of $LM$. We consider $n$-symplectic geometry on the principal bundle $LM$ to be a covering theory in the sense that for any $1 \leq p < n$ we obtain a "quotient" symplectic-type theory on an associated bundle whose structure is a $(p + 1)$-form. Of particular interest are two specific examples pertaining to the classical mechanics of particles and of fields, respectively.

## 4. Symplectic geometry on $T^*M$ derived from $n$-symplectic geometry on $LM$

For our first example, we review a result previously obtained by Norris [12]. Let $p = 1$ and observe that $\wedge^1 M = T^*M$, the standard symplectic manifold for classical particle mechanics. The identification of $T^*M$ with an associated bundle, $T^*M \simeq LM \times_{GL(n,\mathbb{R})} \mathbb{R}^{n*}$, motivated Norris to establish that the canonical one-form $\vartheta$ on $T^*M$ "has its roots

in a more basic structure on $LM$" [12, p.52], namely, the $\mathbb{R}^n$-valued soldering one-form $\theta$ on $LM$. More recently Norris credits [12] Jedrzej Śniatycki for observing the exact relationship, the $p = 1$ case of Eq. (9), between the canonical one-form $\vartheta$ on $T^*M$ and the soldering one-form $\theta$ on $LM$. Thus the fundamental building block $\vartheta$ for canonical symplectic geometry on $T^*M$ is induced from the soldering one-form $\theta$ on $LM$.

In this example, Norris [12] shows that the homogeneous polynomial observables and their corresponding Hamiltonian vector fields on $T^*M$ are induced from related objects on $LM$. In particular, elements of $ST^q$ induce $q$th degree homogeneous polynomial observables on $T^*M$ as follows. Consider $T^*M$ as the associated bundle $LM \times_{GL(n,\mathbb{R})} \mathbb{R}^{n*}$. Then for $\hat{f} \in ST^q$ define $\tilde{f} : T^*M \to \mathbb{R}$ by

$$\tilde{f}([u,\alpha]) := \langle \hat{f}(u), \overbrace{(\alpha, \ldots, \alpha)}^{q} \rangle, \tag{12}$$

where $[u, \alpha] \in T^*M$, $u = (x, e_i) \in LM$, and the brackets denote the evaluation of $\hat{f}(u) \in (\otimes_s)^q \mathbb{R}^n$ at $q$ points in $\mathbb{R}^{n*}$. The tensorial character of $\hat{f}$ guarantees that this definition is independent of choice of representatives of the equivalence class $[u, \alpha]$. In local coordinates, $\pi_j^i(x, e_k)\alpha_i = e^i(\partial/\partial x^j)\alpha_i = p_j(e^i\alpha_i)$ where $\{p_j\}$ are the standard momentum coordinates on $T^*M$ defined by the local chart $\{x^i\}$ on $M$. Then, for example, for $q = 2$ take $\hat{f} = \hat{f}^{ij}r_i \otimes_s r_j$ where $\hat{f}^{ij} = f^{ab}(x)\pi_a^i\pi_b^j$. The definition in (12) yields

$$\tilde{f} = f^{ab}(x)p_a p_b, \tag{13}$$

which is a homogeneous quadratic polynomial observable on $T^*M$. Furthermore, the equivalence class of Hamiltonian vector fields $[\![X_{\hat{f}}]\!]$ for $\hat{f} \in ST^q$ may be mapped to the Hamiltonian vector field $X_{\tilde{f}}$ of $\tilde{f}$ on $T^*M$, where $\tilde{f}$ is induced from $\hat{f}$ as in (12). In this $p = 1$ case the map (10) reduces to $\phi_\alpha(u) = [u, \alpha] \in T^*M$ where $\alpha$ is any nonzero element of $\mathbb{R}^{n*}$.

**Theorem 3.** *Let $\hat{f} \in ST^q$, let $[\![X_{\hat{f}}]\!]$ be the associated equivalence class of Hamiltonian vector fields determined by (3), let $X_{\hat{f}}^{i_1 i_2 \cdots i_{q-1}}$ denote any set of representatives of $[\![X_{\hat{f}}]\!]$, and let $\tilde{f}$ be the degree $q$ homogeneous polynomial observable on $T^*M$ determined by $\hat{f}$ as in (12). Then, for $\alpha \neq 0$,*

$$X = q!\phi_{\alpha*}(X_{\hat{f}}^{i_1 i_2 \cdots i_{q-1}}\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_{q-1}})$$

*is a well-defined vector field on $T^*M$ with the 0-section deleted, and $X = X_{\tilde{f}}$.*

Hence the canonical symplectic geometry of homogeneous polynomial observables on $T^*M$ is induced from the $n$-symplectic geometry on $LM$.

## 5. Multisymplectic geometry on $Z$ derived from $(n + k)$-symplectic geometry on $L_V Y$

For our second example, we study the extension of the theory of symplectic geometry to a multivariable version describing classical field theories. Early attempts to define a

manifold with such a *multisymplectic* geometry consisted of constructions of linear vector bundles [6,7] on the total space of a fiber bundle $Y \rightarrow M$, where $Y$ is the field configuration space of the field theory. The multi-author monograph GIMMsy [5] and papers by Gotay [3,4] discuss an affine version of multisymplectic geometry, in which the "phase space" bundle is the bundle of affine cojets $J^*Y$, the vector bundle over $Y$ whose fiber at $y$ is the set of affine maps from $J_y Y$ to $\wedge^n_{\pi(y)} M$. This is considered to be the affine density-valued dual space to the jet bundle $JY$ over $Y$. The fiber dimension of $J^*Y$ over $Y$ is $nk + 1$.

The vector bundle of affine cojets $J^*Y$ is the best model to date for multisymplectic geometry because there exists a canonical form on $J^*Y$, analogous to the canonical symplectic potential one-form on $T^*M$. For convenience, we work with another description of $J^*Y$. GIMMsy establishes a canonical isomorphism from $J^*Y$ to a sub-bundle $Z$ of the bundle of $n$-forms of $Y$, defined fiberwise over $Y$ by

$$Z_y := \{z \in \wedge^n_y Y \mid v \lrcorner w \lrcorner z = 0 \ \forall v, w \in V_y\},$$

where $V_y$ is the subspace of vertical vectors in $T_y Y$. The *canonical multisymplectic n-form* $^n\Theta$ on $Z$ is defined by $^n\Theta(z) := \pi^*_{YZ}(z)$. In contrast to the linear multisymplectic theories [6,7], $^n\Theta$ exists independently of additional hypotheses, such as choice of Ehresmann connection.

Now we construct the appropriate principal bundle formulation of the multivariable symplectic geometry for affine field theories, analogous to the formulation of the frame bundle $LM$ for standard symplectic geometry on $T^*M$. Applying Norris's theory of $n$-symplectic geometry [11] to an arbitrary fiber bundle $Y$ over $M$ with fiber dimension $k$, we produce the linear frame bundle $LY$ over $Y$ as a principal fiber bundle with structure group $GL(n+k, \mathbb{R})$ and $(n + k)$-symplectic potential $\theta$. Now define the *vertically adapted frame bundle* $L_V Y$ to be the reduced sub-bundle of $LY$ defined by

$$L_V Y := \{(y, \{e_i, \epsilon_A\}) \mid y \in Y, \{e_i, \epsilon_A\} \text{ is a frame of } T_y Y, \{\epsilon_A\} \text{ is a frame of } V_y\}.$$

The structure group $G_A$ of $L_V Y$ is a semidirect product of $G := GL(n) \times GL(k)$ and $\mathbb{R}^{k \times n}$.

In order to construct the tensor bundle $\wedge^p Y$ as a bundle associated to $L_V Y$, define the linear left action of $G_A$ on the vector space $(\otimes_a)^p \mathbb{R}^{n+k^*}$ to be the restriction of the natural action of $GL(n + k, \mathbb{R})$ on $(\otimes_a)^p \mathbb{R}^{n+k^*}$ to the subgroup $G_A \subset GL(n + k, \mathbb{R})$. The vector bundle $\wedge^p Y$ is then identified with $L_V Y \times_{G_A} (\otimes_a)^p \mathbb{R}^{n+k^*}$.

We shall now restrict ourselves to $p = n$. Let $(r_\mu)$, $\mu = 1, \ldots, n+k$, be the standard basis of $\mathbb{R}^{n+k}$ and let $(r^\mu)$ be the corresponding dual basis. For convenience, define $R_{\mu_1 \cdots \mu_n} := r_{\mu_1} \wedge \cdots \wedge r_{\mu_n}$ and $R^{\mu_1 \cdots \mu_n} := r^{\mu_1} \wedge \cdots \wedge r^{\mu_n}$. Then the *Levi-Civita tensor* $\epsilon \in (\otimes_a)^n \mathbb{R}^{n+k^*}$ is represented by $\epsilon := \epsilon_{i_1 \cdots i_n} R^{i_1 \cdots i_n}$ where $\epsilon_{i_1 \cdots i_n} = \text{sgn}\,\sigma$ and $\sigma$ is the permutation of $n$ elements expressed by $(\sigma(1), \ldots, \sigma(n)) = (i_1, \ldots, i_n)$. If $n = 1$ and $(B, \lambda) \in \mathbb{R}^k \times \mathbb{R}$, then define $V(B, \lambda) \in (\otimes_a)^n \mathbb{R}^{n+k^*}$ by components $V_i(B, \lambda) = \lambda$ and $V_A(B, \lambda) = B_A$. If $n > 1$ and $(B, \lambda) \in \mathbb{R}^{n \times k} \times \mathbb{R}$, then define $V(B, \lambda) \in (\otimes_a)^n \mathbb{R}^{n+k^*}$ by components

$$V_{i_1 \cdots i_n}(B, \lambda) = (1/n!)\lambda \epsilon_{i_1 \cdots i_n},$$

$$V_{A i_1 \cdots i_{n-1}}(B, \lambda) = (1/n!) B^j_A \epsilon_{j i_1 \cdots i_{n-1}}$$

and

$$V_{A_1...A_l i_1...i_{n-l}}(B, \lambda) = 0 \quad \forall l \geq 2.$$

All other components are defined to be identically zero. The map

$$\mathbb{R}^{n \times k} \times \mathbb{R} \to (\otimes_a)^n \mathbb{R}^{n+k^*} : (B, \lambda) \mapsto V(B, \lambda)$$

is a linear injection. The $G_A$-action on $\mathbb{R}^{n \times k} \times \mathbb{R}$ defined by

$$\begin{pmatrix} N & 0 \\ A & K \end{pmatrix} \cdot (B, \lambda) := \det(N^{-1})(NBK^{-1}, \lambda - \text{tr}(BK^{-1}A))$$

induces a monomorphism of vector bundles $L_\vee Y \times_{G_A} (\mathbb{R}^{n \times k} \times \mathbb{R}) \to L_\vee Y \times_{G_A} ((\otimes_a)^n \mathbb{R}^{n+k^*})$. So we now have a description of $Z$ as a vector bundle associated to $L_\vee Y$. Indeed, $L_\vee Y \times_{G_A} (\mathbb{R}^{n \times k} \times \mathbb{R})$ is a vector bundle over $Y$ with fiber dimension $nk+1$. It has a canonical $n$-form obtained from pulling back the canonical $n$-form on $L_\vee Y \times_{G_A} (\otimes_a)^n \mathbb{R}^{n+k} \simeq \wedge^n Y$.

Let $i : L_\vee Y \hookrightarrow LY$ be the inclusion map. The $(n + k)$-*symplectic potential* on $L_\vee Y$ is the $\mathbb{R}^{n+k}$-valued one-form $i^*\theta$. We may establish the relationship between the $(n + k)$-symplectic potential $i^*\theta$ and the multivariable symplectic potential $^n\Theta$ on $Z$, which follows from the $p = n$ case of Eq. (9). Now observe that the map

$$\phi_{(B,\lambda)} : L_\vee Y \to Z : w \mapsto [w, (B, \lambda)]$$

is fiber preserving over $Y$ (see Eq. (10)). If rank $B = r$, then $\phi_{(B,\lambda)}$ has for its range a sub-bundle of $Z$ each fiber of which is isomorphic to the set of rank $r$ matrices in $\mathbb{R}^{n \times k}$. Then the potentials may be related by a special case of Eq. (11):

$$\phi^*_{(B,\lambda)}(^n\Theta) = \langle \wedge^n i^*\theta, V(B, \lambda) \rangle. \tag{14}$$


## 6. Momentum observables on $L_\vee Y$ and $Z$

Now we shall proceed to develop the relationships between the algebras of observables and Hamiltonian vector fields on $L_\vee Y$ and the analogous algebras on $Z$. We shall demonstrate that the difficulties with the algebra of observables on $Z$ is not present in the analogous algebra on $L_\vee Y$.

Following GIMMsy, let $v$ be a projectable vector field on the total space $\pi : Y \to M$ and define a *momentum observable* [5] based on $v$ to be an $(n - 1)$-form $f_v$ on $Z$ given by $f_v(z) := \pi^*_{YZ}(v \lrcorner z)$. Assign to $f_v$ a Hamiltonian vector field $X_{f_v}$ via the multisymplectic structure equation,

$$\mathrm{d}f_v = -X_{f_v} \lrcorner \mathrm{d}(^n\Theta). \tag{15}$$

A problem with this geometrical structure is that the naturally defined "Poisson" bracket of two momentum observables,

$$\{f_v, f_w\} := -X_{f_v} \lrcorner X_{f_w} \lrcorner \mathrm{d}(^n\Theta), \tag{16}$$

results in another momentum observable only up to the addition of an exact $(n - 1)$-form. More precisely,

$$\{f_v, f_w\} := f_{[v,w]} + d(X_{f_v} \lrcorner X_{f_w} \lrcorner^n \Theta). \tag{17}$$

Although the vector field $[v, w]$ is projectable, the set of momentum observables does not form a Lie algebra under the bracket in (16), which is otherwise analogous to the Poisson bracket on $C^\infty(T^*M)$ in the standard theory of symplectic geometry. If we extend to the space of momentum observables plus exact $(n - 1)$-forms and quotient by the exact $(n - 1)$-forms, we would form a Lie algebra under the obvious extension of the bracket in (16). However, this Lie algebra is devoid of the observables analogous to the position observables in the standard theory.

We recall (see the end of Section 2) that there is a one–one correspondence between vector fields on $Y$ and tensorial $\mathbb{R}^{n+k}$-valued functions on $LY$. By restriction, there is a one–one correspondence between vector fields on $Y$ and $G_A$-tensorial $\mathbb{R}^{n+k}$-valued functions on $L_V Y$. We define a vector subspace of $T^1(L_V Y)$, the space of $G_A$-tensorial $\mathbb{R}^{n+k}$-valued functions on $L_V Y$, as follows. Define $\mathcal{X}_{\text{Proj}} Y$ to be the space of vector fields on $Y$ that are smoothly projectable to $M$, and define $T_V^1(L_V Y)$ to be the subspace of $T^1(L_V Y)$ that corresponds only with $\mathcal{X}_{\text{Proj}} Y$.

The $(n + k)$-symplectic structure on $L_V Y$ is the nondegenerate $\mathbb{R}^{n+k}$-valued two-form $i^* d\theta$. The $(n + k)$-symplectic structure equation for $L_V Y$ is (cf. Eq. (6))

$$d\hat{f} = -X_{\hat{f}} \lrcorner i^* d\theta, \tag{18}$$

where $X_{\hat{f}}$ is a vector field on $L_V Y$ and $\hat{f}$ is a smooth $\mathbb{R}^{n+k}$-valued function on $L_V Y$. As in the $n$-symplectic theory on $LM$, the structure equation admits neither all vector fields nor all $\mathbb{R}^{n+k}$-valued functions.

**Proposition 1.** *The only elements of $T^1(L_V Y)$ that are admissible in the $(n+k)$-symplectic equation are those in the subspace $T_V^1(L_V Y)$. Moreover, the subspace $T_V^1(L_V Y)$ forms a Lie algebra under the bracket defined in Eq. (4).*

Recalling that we constructed observables on both bundles $L_V Y$ and $Z$ from $\mathcal{X}_{\text{Proj}} Y$, we see as an immediate consequence that the vector space of observables $T_V^1(L_V Y)$ is in one–one correspondence with the vector space of momentum observables on $Z$. In addition, the Hamiltonian vector fields on the two bundles may also be related.

**Theorem 4.** *Let $v \in \mathcal{X}_{\text{Proj}} Y$, let $\hat{f}_v \in T_V^1(L_V Y)$ be obtained from $v$ and let $X_{\hat{f}_v}$ be the Hamiltonian vector field on $L_V Y$ obtained from $\hat{f}_v$. Then $\phi_{(B,\lambda)*} X_{\hat{f}_v} = \tilde{X}_{f_v}$ where the momentum observable $f_v$ is obtained from $v$ and $\tilde{X}_{f_v}$ is the corresponding Hamiltonian vector field on the stratum of $Z$ defined by $(B, \lambda)$.*

We now seek to explain in terms of phenomena on $L_V Y$ the problem with the attempted algebraic structure on the space of momentum observables on $Z$. To this end we introduce a

version of the multisymplectic geometry of $Z$ on $L_V Y$. Define a representation of $T_V^1(L_V Y)$ into the space of $(\otimes_a)^n \mathbb{R}^{n+k}$-valued $(n-1)$-forms by

$$\hat{f} \mapsto \hat{f} \wedge (\wedge^{n-1} i^* \theta). \tag{19}$$

The form $d(\wedge^n i^* \theta)$ is both closed and nondegenerate, so it may serve as a new type of generalized symplectic $(n+1)$-form on $L_V Y$. In particular, for each $(\otimes_a)^n \mathbb{R}^{n+k}$-valued $(n-1)$-form of the type $\hat{f} \wedge (\wedge^{n-1} i^* \theta)$ where $\hat{f} \in T_V^1(L_V Y)$, we may assign a Hamiltonian vector field $X_{\hat{f} \wedge (\wedge^{n-1} i^* \theta)}$ via the new structure equation containing $d(\wedge^n i^* \theta)$, namely the equation

$$d(\hat{f} \wedge (\wedge^{n-1} i^* \theta)) = -(1/n) X_{\hat{f} \wedge (\wedge^{n-1} i^* \theta)} \lrcorner d(\wedge^n i^* \theta).$$

We may easily verify that $X_{\hat{f} \wedge (\wedge^{n-1} i^* \theta)} = X_{\hat{f}}$, so that in fact no new "vector field" information is provided by this structure equation beyond what is contained in the original $(n+k)$-symplectic structure equation (Eq. (18)). The natural choice for a definition of bracket for the new "momentum observables" is

$$\{\hat{f} \wedge (\wedge^{n-1} i^* \theta), \hat{g} \wedge (\wedge^{n-1} i^* \theta)\} := -X_{\hat{f}} \lrcorner (X_{\hat{g}} \lrcorner (i^* d\theta \wedge (\wedge^{n-1} i^* \theta)))$$

$$= X_{\hat{f}} \lrcorner d\hat{g} \wedge (\wedge^{n-1} i^* \theta). \tag{20}$$

However, one finds that the bracket does not close on the set of $(n-1)$-forms of the type $\hat{f} \wedge (\wedge^{n-1} i^* \theta)$ where $\hat{f} \in T_V^1(L_V Y)$. In fact one finds that

$$\{\hat{f} \wedge (\wedge^{n-1} i^* \theta), \hat{g} \wedge (\wedge^{n-1} i^* \theta)\}$$

$$= \{\hat{f}, \hat{g}\} \wedge (\wedge^{n-1} i^* \theta) + (n-1) d(\hat{f} \wedge \hat{g} \wedge (\wedge^{n-2} i^* \theta)). \tag{21}$$

This equation should be compared to Eq. (17). Thus we have found the analogue of the problem of closing the Poisson-type bracket on momentum observables defined on $Z$.

## 7. Conclusions

Standard symplectic geometry on the cotangent bundle $T^* M$ to an $n$-dimensional manifold $M$ can be derived from $n$-symplectic geometry on $LM$. This is exactly the sense in which $n$-symplectic geometry is a covering theory for the symplectic geometry on $T^* M$. If $T^* M$ is the prototypical symplectic manifold for classical particle mechanics then the vector bundle $Z \simeq J^* Y$ of affine cojets of a fiber bundle $Y$ is the prototypical multisymplectic manifold for classical field theories. Our current investigation has reinforced this point of view by constructing the bundle of affine cojets as a bundle associated to $L_V Y$, the principal bundle of vertically adapted linear frames. We demonstrated that the bundle $L_V Y$ possesses a generalized symplectic geometry pulled back from the geometry of the full frame bundle $LY$, and that the $(n+k)$-symplectic geometry on $L_V Y$ "covers" the multisymplectic geometry on $Z$.

Following the GIMMsy [5] monograph, one may define momentum observables on $Z$ from projectable vector fields on $Y$ and then assign Hamiltonian vector fields using the

multivariable symplectic structure equation. The problem with the GIMMsy construction is that the naturally defined "Poisson" bracket of two such momentum observables, as defined by GIMMsy, is another momentum observable only up to the addition of an exact form. Thus the set of momentum observables does not form a Lie algebra under this "Poisson" bracket. We duplicated this result by introducing a version of the multisymplectic geometry of $Z$ on $L_V Y$. Indeed, the representation defined in (19) is faithful and the failure of the "Poisson" bracket to close on the set of momentum observables is mirrored by Eq. (21).

The principal bundle geometry on $L_V Y$ not only reconstructs the above problem, but also suggests a remedy. In $(n + k)$-symplectic geometry one assigns to each projectable vector field on $Y$ an $\mathbb{R}^{n+k}$-valued tensorial function on $L_V Y$ and an associated Hamiltonian vector field defined by the $(n+k)$-symplectic structure equation. We know that the set of all such tensorial functions forms a Lie algebra under the Poisson bracket defined in Eq. (4). Thus if it is only the Hamiltonian vector field that is important, then we have three ways to find essentially the same vector field, and any one seems to be as good as either of the other two. The three methods are: (1) multisymplectic geometry on $Z$; (2) the version of multisymplectic geometry of $Z$ defined on $L_V Y$; and (3) $(n + k)$-symplectic geometry on $L_V Y$. However, to obtain an algebra of observables there is only one clear theory which leads to a consistent, well-defined Lie algebra, namely the $(n + k)$-symplectic theory.

This investigation is foundational in the sense that in order to study classical particles or fields, we must understand the origin and nature of the observables. Of course, this can be done without the covering theory on the principal bundle. However, for example, in covariant field theory we must witness events from a preferred reference frame. Here the principal bundle is useful because it describes the space of all reference frames, and sections of the principal bundle determine preferential frames of inertial observers.

There are implications of these results in the application of the procedure to geometric quantization of field theories. If one wants a Lie algebra to use in a geometric quantization program, then only the $(n + k)$-symplectic method will work without extension of the class of observables. An elaboration of some of the above results is found in [2,9], and further developments as well as potential applications will be discussed in greater detail in future publications.

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